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TECHNICAL NOTE

No. 1401

INTRODUCTION TO THE PROBLEM OF ROCKET-POWERED AIRCRAFT PERFORMANCE

By H. Reese Ivey, Edward N. Bowen, Jr., and Lester F. Oborny

Langley Memorial Aeronautical Laboratory
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Page 17: Equation (28) should be corrected as follows:

$$\left(\frac{A_t}{A_e}\right)^2 = \left[\left(\frac{P_e}{P_c}\right)^{\frac{2}{\gamma}} - \left(\frac{P_e}{P_c}\right)^{\frac{\gamma+1}{\gamma}}\right] \frac{(\gamma + 1)^{\frac{\gamma+1}{\gamma-1}}}{(2)^{\frac{2}{\gamma-1}}(\gamma - 1)} \tag{28}$$

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SUMMARY

An introduction to the problem of determining the fundamental limitations on the performance possibilities of rocket-powered aircraft is presented. Previous material on the subject is reviewed and given in condensed form along with supplementary analyses.

Some of the problems discussed are:

(1) Limiting velocity of a rocket projectile
(2) Limiting velocity of a rocket jet
(3) Jet efficiency
(4) Nozzle characteristics
(5) Maximum attainable altitudes
(6) Range

Formulas are presented relating the performance of a rocket-powered aircraft to basic weight and nozzle dimensional parameters. The use of these formulas is illustrated by their application to the special case of a nonlifting rocket projectile.

INTRODUCTION

Rocket engines carry both the fuel and the oxidizing agent needed to create thrust and therefore are not limited to the atmosphere for their operation. Since a rocket engine develops high thrust at extreme altitudes, rocket-powered aircraft can be expected to attain very high speeds in the low density upper atmosphere. Because of the unusual characteristics of the engine and aircraft operating conditions, an extensive analysis is needed to determine the optimum aircraft configuration and flight plan for attaining maximum possible performance. Before such an analysis can be conducted in detail, the fundamental limitations of rocket performance - such as variation of thrust with altitude, variation of propulsive efficiency with speed, and the possible range of altitude and velocity that may be encountered - must be determined.

The purpose of this paper is to present the relations among the following variables: (1) performance (range, altitude, speed, and so forth), (2) fuel characteristics, (3) fuel loads, (4) aircraft weight, and (5) nozzle dimensions. The application of the fundamental formulas to determine the optimum configuration and performance of actual aircraft is not included. However, the optimum performance of a simple aerodynamic shape such as a projectile is used to illustrate the methods developed.

A large amount of work has already been done in developing the theory of rocket projectiles; however, the reports are not available as references for the present paper because of their classification. This work is reviewed in condensed form, and additions are made along the line of performance limitation and operating efficiency.

The development of the fundamental equations involves much mathematical manipulation which is not required for the presentation of the final results of the analysis. The present paper is therefore divided into two parts: The first part gives a nonmathematical discussion, which is complete in itself, for the reader not concerned with the derivation of the formulas involved; the second part presents the mathematical derivations and assumptions used in this analysis.

SYMBOLS

The following symbols are used in the section entitled "Technical Discussion and Derivation of Equations."

A projected frontal area

a speed of sound or acceleration

C constant defined as $r_oV_o \cos \theta$

C_D drag coefficient based on frontal area

D drag

E total energy

F force

G universal constant of gravitation

g acceleration due to earth's force of gravity

H heating value of fuel

J mechanical equivalent of heat

h altitude

K constant defined as $\frac{T}{m_o}$

m mass

M mass of the earth

P pressure

Q parameter defined as $\frac{r_o V_o^2}{GM}$

R range

r distance from center of earth

S surface area

T thrust

t time

V velocity

W weight

γ ratio of specific heats

η efficiency

ρ air density

ϕ in polar coordinates, angle measured from positive X-axis

θ launching angle

Subscripts:

a	atmospheric
B	during or at end of burning
c	combustion chamber; during coasting
e	empty; exit
f	final; fuel
j	jet
o	initial; at surface of earth
R	remaining fuel
T	projected in thrust direction
t	throat
av	average
eff	effective
inst	instantaneous
max	maximum
mix	mixture
opt	optimum
SL	sea level
ult	ultimate
vac	vacuum

GENERAL DISCUSSION AND PRESENTATION OF RESULTS

Principle of Operation

A rocket carries all the fuel and oxidizing agent required to operate the engine and hence the engine may operate both in air and in a vacuum. The rocket engine exerts a forward thrust as a reaction to expelling the exhaust products rearward at high velocity as a jet. This thrust can be expressed in terms of effective exhaust jet velocity and fuel consumption by equating thrust to rate of change of the momentum of the exhaust gases being expended. Figure 1 shows the variation of fuel consumption required per unit of thrust with effective jet velocity. It is apparent that high jet velocities are required for low specific fuel consumption.

Rocket Velocity Limitation

The final velocity of a rocket-powered aircraft is dependent upon air resistance, effective jet velocity, and percentage of fuel load. Even in the absence of air resistance a limit is set by the fuel load and the effective jet velocity on the final speed that may be obtained. The maximum velocity attained by a rocket-powered aircraft in an empty gravity-free space may be determined by equating the thrust to the product of instantaneous rocket mass and instantaneous rocket acceleration. Figure 2 presents this limiting velocity as a function of both fuel load and jet velocity and shows that the speed limitation is directly proportional to jet velocity and is critically dependent on fuel load. For instance, at a 90-percent fuel load the value of limiting velocity is 3.5 times as large as the value at a 45-percent fuel load.

Jet Velocity Limitation

The maximum possible jet velocity for a specific fuel would be obtained when all the heat released during combustion of the fuel was converted to exhaust jet kinetic energy. In this case the jet velocity would vary as the square root of the heating value of the fuel. Actual jet velocities are further limited by effects of radiation, heat capacity lag, dissociation, nozzle characteristics, and combustion efficiency. Jet velocity alone is insufficient basis for selection of the best fuel; handling qualities, fuel density, and availability must also be considered.

Jet Efficiency

Since for constant burning rate the thrust of a rocket engine is dependent only upon the velocity of the jet relative to the engine, a rocket develops constant thrust at all rocket velocities in a vacuum. Rocket speeds in excess of jet speeds are therefore possible. (See fig. 2.) The manner in which the jet efficiency varies as the speed of the rocket increases may be easily determined.

The rate of expending fuel can be considered constant. When the rocket is at rest, all the fuel energy released during burning appears as kinetic energy of the jet, and hence none of the energy is used to increase the energy of the rocket, and the jet efficiency is zero at zero forward speed. When the rocket is moving at jet speed, the exhaust gases remain stationary and have no energy relative to the earth. Then all the usable fuel energy is being used to increase the kinetic energy of the rocket, and the jet efficiency is 100 percent. When the rocket is moving at twice jet speed, the exhaust gases are ejected at jet speed relative to the ground, and hence all the fuel energy released during burning again goes into the jet. The jet efficiency therefore returns to zero at twice jet velocity. (See fig. 3.) The fact that the efficiency is zero at twice jet speed does not mean that the rocket cannot continue to accelerate, but simply means that any additional increase in speed above twice jet speed will result in a decrease in kinetic energy due to the steady decrease in rocket mass caused by fuel consumption.

A similar method of analysis can be used to determine the average jet efficiency over the period of a complete flight. Figure 4 shows that this average jet efficiency cannot exceed 65 percent.

Nozzle Parameters

The thrust of a rocket engine is equal to the integral of the surface pressure times the projected area in the thrust direction for the inner and outer surfaces of the rocket as illustrated by figures 5 to 7. Nozzle effectiveness can be defined as the ratio of the thrust actually produced to the thrust that would be available by perfect conversion of the fuel energy to thrust. Nozzle dimensions can be related to nozzle effectiveness by use of the laws of conservation of energy and conservation of momentum. This relationship is presented graphically in figure 8 which may be useful in estimating the correct nozzle dimensions for optimum nozzle operation with various ratios of atmospheric pressure to combustion chamber pressure. Figure 8 indicates that low ratios of atmospheric pressure to combustion chamber pressure in combination with high ratios of nozzle exit area to throat area are needed for reasonable values of nozzle effectiveness. In actual practice a compromise between nozzle structural considerations and nozzle effectiveness is necessary. It is obvious that for a given constant combustion chamber pressure, the effectiveness of a nozzle of given dimensions will increase with an increase in altitude.

Maximum Altitude

The maximum altitude that can be attained by a rocket launched vertically from the ground is determined by the precentage of fuel load, the initial acceleration, the jet velocity, and the air drag. The basic case of a rocket operating in a vacuum is considered first and then the effect of air resistance on the altitude attained is discussed.

Figure 9 shows the velocity reached at the end of burning for rocket projectiles fired vertically in a vacuum for an effective jet velocity of 8000 feet per second, and figure 10 shows the altitude reached at the end of burning for the same projectiles. At the end of burning, the rockets continue to increase in altitude until the total kinetic energy of the rocket at the end of burning is converted to potential energy in the form of increased altitude. Figure 11 shows the maximum altitudes for the rockets investigated. It is evident that large fuel loads and high initial accelerations are required for reaching very high burnt velocities or very high maximum altitudes.

If a drag factor (fig. 12) is assumed for the body shown in figure 13, the resulting variation of maximum altitude attained with initial acceleration and fuel load can be represented by figure 13 for a jet velocity of 8000 feet per second. It should be noted that an increase in the size of a rocket decreases the drag per unit volume and, therefore, very large rockets will exceed the performance given in figure 13. This figure indicates that there exists an optimum initial acceleration for reaching maximum altitude that is almost the same for all moderate fuel loads at a value of the ratio of thrust to weight equal to 3 (that is, initial acceleration equals twice that of gravity).

Figure 13 shows that for rockets which have ratios of thrust to initial weight of 100 there is an optimum fuel load of approximately 60 percent. Rockets which have fuel loads over 60 percent have such light empty weights that they decelerate rapidly after burning and do not go so high as those with heavier empty weights.

The drag variation used in connection with figures 12 and 13 is greatly simplified and would have to be a function of Reynolds number for a more exact study. The introduction of drag as a function of Reynolds number would, in general, shift the curves of figure 13 and change the crossings of the curves.

The equation for maximum altitude can be set equal to infinity and solved for the fuel loads necessary to leave the gravitational field of the earth. A rocket moving 25,000 miles per hour and shot at any angle above the horizontal has enough energy to escape from the earth. Figure 14 shows a graph of the fuel loads required to reach this speed in a vacuum.

The large fuel loads required for escape velocities may be impractical from structural considerations. It may be necessary, therefore, to boost the final rocket by means of expendable booster stages. If the ratio of the weight of the fuel used in the booster stage to the initial weight of the combination rocket is equal to the ratio of the fuel in the final rocket to the initial weight of the final rocket, then the burnt velocity of the final rocket is twice that which would be possible with the final rocket alone. Three-stage rockets operate on a similar principle. Figure 14 points out the large savings in required fuel load that are possible with two-stage and three-stage rockets.

When the required fuel loads are high, the ratio of weights of one stage of the rocket to the next is large.

Maximum Range

The flight path of a rocket fired in a vacuum with one component of velocity parallel to the surface of the earth is an ellipse with one focus at the center of the earth, provided the speed of the rocket is less than escape speed. If it is assumed that the rocket is launched with a given velocity at the surface of the earth and receives no additional thrust and that the laws of conservation of energy and conservation of angular momentum apply, there is obtained figure 15 which shows the variation of range with launching velocity and launching angle. This figure indicates that short-range rockets should be launched at an angle of approximately 45° but long-range rockets should be launched almost parallel to the ground. In practice the angle must be corrected to allow for the effects of the atmosphere.

Figure 16 shows the maximum range attainable as a function of the launching velocity for rockets launched at the optimum angle. The range is shown to increase very rapidly with increasing speed.

The maximum altitude reached by the rocket when fired at the angle for maximum range is shown in figure 17. The rockets fired at approximately 16,000 miles per hour reach higher altitudes than any others. The range that corresponds to this launching velocity is 6000 miles. (See fig. 16.) Rockets with either shorter or longer ranges do not reach as high an altitude.

The fuel loads required for attaining various ranges are given for one-stage, two-stage, and three-stage rockets in figure 18. This figure indicates that multistage rockets are necessary for attaining long ranges with moderate fuel loads.

TECHNICAL DISCUSSION AND DERIVATION OF EQUATIONS

Principle of Operation

A rocket carries all the fuel and oxidizing agent required to operate the engine, hence the engine may operate both in air and in a vacuum.

The rocket engine exerts a forward thrust as a reaction to expelling the exhaust products rearward at high velocity as a jet. This thrust can be expressed in terms of exhaust jet velocity and fuel consumption by equating the thrust T to the rate of change of momentum of the exhaust gases being expended

$$T = \frac{V_j}{g} \frac{dW}{dt} \tag{1}$$

where g is the acceleration due to gravity, V_j is the effective velocity of the exhaust jet relative to the rocket, and dW/dt is the rate of fuel consumption (numerically equal to rate of change of rocket weight).

Example 1. Find the fuel consumption required to produce 1000 pounds of thrust with a jet velocity of 6000 feet per second.

From equation (1)

$$\frac{dW}{dt} = \frac{Tg}{V_j} = \frac{1000 \times 32.2}{6000}$$

$$= 5.37 \text{ pounds per second}$$

This example emphasizes the very high fuel consumption required by rocket engines. Figure 1 shows the variation of specific fuel consumption with effective jet velocity.

Rocket Velocity Limitation

The velocity of a rocket depends upon many factors, such as jet velocity, fuel consumption, and air resistance; however, certain important relations can be obtained when some of the factors are neglected. If the effects of external forces such as gravity and air resistance are neglected, the thrust equals the instantaneous mass of the rocket times the acceleration

$$T = \frac{W_o - t\frac{dW}{dt}}{g} \frac{dV}{dt} \tag{2}$$

where t is the time the rocket has been burning. Substituting the value of T from equation (1) in equation (2) gives

$$\frac{V_j}{g}\frac{dW}{dt} = \frac{W_o - t\frac{dW}{dt}}{g}\frac{dV}{dt}$$

Separating the variables $\left(\frac{dW}{dt}\text{ can be assumed constant}\right)$ and integrating gives

$$V = V_j \log_e \frac{W_o}{W_o - t\frac{dW}{dt}} + V_o$$

The final speed of the rocket $\left(\text{when } W_o - t\frac{dW}{dt} = W_e\right)$ is then

$$V_f = V_j \log_e \frac{W_o}{W_e} + V_o \tag{3}$$

where W_e is the empty weight.

This equation can also be put in the form:

$$V_f = V_j \log_e \frac{1}{1 - \frac{W_f}{W_o}} + V_o \tag{4}$$

where W_f/W_o is the fractional part of the initial weight which consisted of fuel. The final velocity of rockets starting from rest is presented in figure 2 as a function of percentage of fuel load and jet velocity by means of equation (4).

Example 2. Estimate the limiting final speed of a rocket having 60 percent of its weight in fuel and having a jet velocity of 6000 feet per second. Figure 2 gives the answer as

$$V_f = 3750 \text{ miles per hour}$$

$$= 5500 \text{ feet per second}$$

Jet Velocity Limitation

The foregoing parts of the present paper have demonstrated that the attainment of high jet velocities is one of the important requirements of rockets. In addition to the various internal efficiencies which act to decrease the actual jet velocity, there is a basic limitation set by the heating value of the fuel. In the limit all the fuel heat of combustion is changed into kinetic energy in the jet. Then the jet velocity is related to the average heating value of a pound of fuel mixture H_{mix}, if the initial heat content of the mixture is not important, by the relation

$$V_j = \sqrt{2gJH_{mix}} = 223.9\sqrt{H_{mix}} \tag{5}$$

Example 3. What is the maximum jet velocity obtainable from complete combustion of a fuel mixture consisting of 78 percent oxygen and 22 percent hydrocarbon? The heating value of the hydrocarbon is 20,750 Btu per pound.

The limiting jet velocity is

$$V_{j_{ult}} = 223.9\sqrt{0.22 \times 20,750}$$
$$= 15,130 \text{ feet per second}$$

In actual practice, radiation, heat capacity lag, dissociation, low operating pressures, incomplete combustion, losses due to viscosity, and other factors combine to cut the effective jet velocity approximately in half.

Jet Efficiency

The sections entitled "Principle of Operation" and "Jet Velocity Limitation" have shown that the thrust developed by a rocket depends upon the jet velocity and fuel rate of burning. For a constant fuel consumption and effective jet velocity, the thrust is essentially constant for all rocket velocities. When the speed of the rocket is zero, the useful work per second done by the jet is obviously zero. As the speed of the rocket is increased, the useful work per second done by the jet increases. The efficiency of the jet in converting jet energy to kinetic energy of the rocket can be expressed by the following ratio:

$$\eta_j = \frac{\text{Rate of change of rocket kinetic energy}}{\text{Rate of kinetic energy (relative to rocket) expelled in jet}}$$

If the kinetic energy of the rocket is

$$\frac{1}{2}mV^2$$

and the time rate of increase of the rocket energy is

$$mV \frac{dV}{dt} - \frac{1}{2} V^2 \frac{dm}{dt}$$

and the kinetic energy per second of the jet relative to the rocket is

$$\frac{1}{2} V_j^2 \frac{dm}{dt}$$

then, η_j becomes

$$\eta_j = \frac{mV \frac{dV}{dt} - \frac{1}{2} V^2 \frac{dm}{dt}}{\frac{1}{2} \frac{dm}{dt} V_j^2} \tag{6}$$

Then, since

$$m = m_o - t \frac{dm}{dt}$$

and, by use of equations (1) and (2)

$$\frac{dV}{dt} = \frac{V_j \frac{dm}{dt}}{m_o - t \frac{dm}{dt}}$$

equation (6) upon simplification becomes

$$\eta_j = \frac{2V_jV - V^2}{V_j^2} = 2\left(\frac{V}{V_j}\right) - \left(\frac{V}{V_j}\right)^2 \tag{7}$$

Figure 3 shows a plot of this equation. The jet efficiency reaches a maximum (100 percent) at a rocket speed equal to the jet speed. Any increase in rocket speed above twice the jet speed results in a net decrease in total rocket kinetic energy. This phenomenon can be explained by the fact that the mass of the rocket decreases (because of fuel consumption) at such a rate that the increase in rocket velocity is insufficient to maintain even constant kinetic energy. The fact that the rocket attains maximum kinetic energy at rocket velocities equal to twice jet velocity leads to the following expression derived from equation (3) for the maximum attainable kinetic energy of a rocket starting from rest and with a fuel load sufficient to accelerate to twice jet velocity:

$$K.E._{max} = \frac{2m_oV_j^2}{e^2}$$

Of additional interest is the average jet efficiency since the rocket was launched, defined as the final kinetic energy of the rocket divided by the kinetic energy expended in the jet.

The kinetic energy of the empty rocket is

$$K.E. = \frac{m_e}{2} V^2$$

and the kinetic energy of the fuel burned was

$$K.E._f = \frac{m_o - m_e}{2} V_j^2$$

Therefore the average jet efficiency is

$$\eta_{j_{av}} = \frac{K.E.}{K.E._f} = \frac{m_e}{m_o - m_e} \frac{V^2}{V_j^2} = \frac{1}{\frac{m_o}{m_e} - 1}\left(\frac{V}{V_j}\right)^2$$

From equation (3)

$$\frac{m_o}{m_e} = e^{V/V_j}$$

Then

$$\eta_{j_{av}} = \frac{1}{e^{V/V_j} - 1}\left(\frac{V}{V_j}\right)^2 \tag{8}$$

Figure 4 shows a plot of equation (8) indicating an approximate maximum jet efficiency of 65 percent for rockets reaching speeds of about 1.6 times jet speed. This figure shows that the average propulsive efficiency of a rocket is not very high, mainly because of the low initial efficiency. Higher average jet efficiencies may possibly be secured by boosting the rocket over the low-speed range by some more efficient means of propulsion.

Nozzle Parameters

In the preceding discussion the rocket characteristics have been frequently expressed in terms of an "effective" jet velocity (defined by equation (1)) in order that the equations would be simplified. The present section will consider in more detail the fundamental nozzle characteristics that influence the effective jet velocity.

The thrust T acting on the rocket is equal to the integral of the pressure P times the projected area in the thrust direction S_T for the inner and outer surfaces of the rocket:

$$T = \int_{\text{Outer}} P \, dS_T + \int_{\text{Inner}} P \, dS_T \tag{9}$$

Figure 5 shows the elemental forces on the outer surface of the rocket at rest in the atmosphere. All the forces balance out except those on the nose with a projected area equal to the exit area of the nozzle. The nose of the rocket has atmospheric pressure on it so that the force due to this pressure is

$$\int_{\text{Outer}} P \, dS_T = -P_a A_e \tag{10}$$

Figure 6 shows the elemental forces on the inner surface of the rocket. The integral of these forces in the thrust direction can be most easily found by considering the force that the inner surface of the rocket exerts on the burning fuel. (See fig. 7.) Equating the resultant force on the gas in the combustion chamber and nozzle to the rate of increase of momentum of the gas gives the expression

$$\Sigma F = \int_{\text{Inner}} P \, dS_T - P_e A_e = V_e \frac{dm}{dt} \tag{11}$$

and hence

$$\int_{\text{Inner}} P \, dS_T = V_e \frac{dm}{dt} + P_e A_e \tag{12}$$

The following equation is obtained by use of equations (10), (12), and (9):

$$T = \left(P_e - P_a\right) A_e + V_e \frac{dm}{dt} \tag{13}$$

The exit pressure for maximum thrust can be found by differentiating equation (13); however, for the sake of having a physical picture of the problem, the optimum exit pressure will be found from other considerations. When a small extension (area = dS_T) is added to the nozzle, the pressures upstream in the nozzle and rocket interior are unchanged since the flow is supersonic. Also, the pressure on the outer surface of the rocket is still atmospheric so that the only changes in the forces on the rocket are those contributed by the pressures on the added nozzle element. Therefore,

$$dT = dS_T \left(P_e - P_a\right)$$

The addition of area to the nozzle obviously increases the thrust as long as $P_e > P_a$ and thrust reaches a maximum when $P_e = P_a$. A rocket at rest, therefore, develops its highest effective jet velocity when the flow is expanded to atmospheric pressure.

The rocket in motion presents a different problem. It is no longer desirable to obtain maximum thrust but rather the best compromise of thrust and drag. If the nozzle is not a factor determining the shape of the rocket, the nozzle should expand the exhaust products to the pressure existing locally on the surface of the rocket around the nozzle exit.

The various terms in equation (13) can be related by the use of a simple one-dimensional nozzle theory derived from Bernoulli's equation

$$\frac{1}{2}\left(V_2^2 - V_1^2\right) = \frac{a_1^2}{\gamma - 1}\left[1 - \left(\frac{\rho_2}{\rho_1}\right)^{\gamma - 1}\right] \tag{14}$$

and the relationship

$$a^2 = \frac{\gamma P}{\rho} \tag{15}$$

If stagnation conditions are assumed to exist in the combustion chamber $\left(V_c = V_1 = 0\right)$, equation (14) becomes

$$\frac{V^2}{a_c^2} = \frac{2}{\gamma - 1}\left[1 - \left(\frac{\rho}{\rho_c}\right)^{\gamma - 1}\right] \tag{16}$$

or

$$\frac{\rho}{\rho_c} = \left(1 - \frac{\gamma - 1}{2}\frac{V^2}{a_c^2}\right)^{\frac{1}{\gamma - 1}} \tag{17}$$

Also

$$\frac{P}{P_c} = \left(1 - \frac{\gamma - 1}{2}\frac{V^2}{a_c^2}\right)^{\frac{\gamma}{\gamma - 1}} \tag{18}$$

and

$$\frac{a^2}{a_c^2} = \left(1 - \frac{\gamma - 1}{2}\frac{V^2}{a_c^2}\right) \tag{19}$$

In the throat of the supersonic nozzle the flow moves with the local speed of sound. From equation (19)

$$V_t = a_t = a_c\sqrt{\frac{2}{\gamma + 1}} \tag{20}$$

Then equations (17) and (20) give density in the throat

$$\rho_t = \rho_c\left(\frac{2}{\gamma + 1}\right)^{\frac{1}{\gamma - 1}} \tag{21}$$

and equations (18) and (20) give pressure in the throat

$$P_t = P_c\left(\frac{2}{\gamma + 1}\right)^{\frac{\gamma}{\gamma - 1}} \tag{22}$$

The massflow (or fuel consumption) is

$$\frac{dm}{dt} = \rho_t A_t V_t = \rho_c A_t a_c\left(\frac{2}{\gamma + 1}\right)^{\frac{\gamma+1}{2(\gamma-1)}} = \rho_e A_e V_e \tag{23}$$

Equation (13) can be written in nondimensional form as

$$\frac{T}{a_c\,\frac{dm}{dt}} = \frac{P_e - P_a}{a_c\,\frac{dm}{dt}}A_e + \frac{V_e}{a_c} \tag{24}$$

By substituting equations (15), (18), and (23) in equation (24)

$$\frac{T}{a_c\,\frac{dm}{dt}} = \left(\frac{P_e}{P_c} - \frac{P_a}{P_c}\right)\frac{A_e}{A_t\gamma\left(\frac{2}{\gamma + 1}\right)^{\frac{\gamma+1}{2(\gamma-1)}}} + \sqrt{\frac{2}{\gamma - 1}\left[1 - \left(\frac{P_e}{P_c}\right)^{\frac{\gamma-1}{\gamma}}\right]} \tag{25}$$

As previously mentioned, the thrust is a maximum when the first term equals zero. The highest value the last term can have is found when $\frac{P_e}{P_c} = \frac{P_a}{P_c} = 0$; that is,

$$\left(\frac{T}{a_c\,\frac{dm}{dt}}\right)_{ult} = \sqrt{\frac{2}{\gamma - 1}} \tag{26}$$

The ultimate jet velocity therefore is

$$\left(\frac{T}{\frac{dm}{dt}}\right)_{ult} = a_c\sqrt{\frac{2}{\gamma - 1}} \tag{27}$$

The effectiveness of a nozzle in producing thrust can be found by dividing equation (25) by equation (26).

The following relation between nozzle area ratio and pressure ratio is obtained by equating mass flow at the nozzle throat to mass flow at the nozzle exit:

$$\left(\frac{A_t}{A_e}\right)^2 = \left[\left(\frac{P_e}{P_c}\right)^{\frac{2}{\gamma}} - \left(\frac{P_e}{P_c}\right)^{\frac{\gamma+1}{\gamma}}\right] \frac{(\gamma + 1)^{\frac{\gamma+1}{\gamma-1}}}{\cancel{2(\gamma - 1)^{\frac{2}{\gamma-1}}}} \quad (2)^{\frac{2}{\gamma-1}} (\gamma - 1) \tag{28}$$

By substituting arbitrary values of P_e/P_c in this equation, simultaneous values of P_e/P_c and A_t/A_e can be obtained for substitution in equation (25) with arbitrary values of P_a/P_c. The results have been plotted in figure 8 which gives the effectiveness of the nozzle in producing thrust. For example: Determine the optimum area ratio to be used on a liquid-fuel rocket operating at sea level (P_a = 14.7) with a combustion chamber pressure of 147 pounds per square inch. The ultimate jet velocity is 15,130 feet per second; therefore, the pressure ratio is

$$\frac{P_a}{P_c} = \frac{14.7}{147} = 0.1$$

Figure 8 gives the peak of the curve for this pressure ratio at an area ratio

$$\frac{A_e}{A_t} = 2.12$$

The ratio of effective jet velocity to ultimate jet velocity is given as

$$\frac{V_j}{V_{j_{ult}}} = 0.607$$

The effective jet velocity as limited by pressure ratio and nozzle dimensions is then

$$V_j = 15{,}130 \times 0.607$$
$$= 9185 \text{ feet per second}$$

The variation of thrust with altitude can be determined from figure 8. For instance, if the rocket of the preceding example is considered to be in a vacuum $\left(\frac{P_a}{P_c} = 0\right)$ and to have the same nozzle $\left(\frac{A_e}{A_t} = 2.12\right)$

$$\frac{V_j}{V_{j_{ult}}} = 0.709$$

or

$$V_j = 10{,}730 \text{ feet per second}$$

From equation (1) it is seen that the thrust is proportional to the effective jet velocity since the mass flow remains constant. The ratio of thrust in a vacuum to thrust at sea level for this particular case is then

$$\frac{T_{vac}}{T_{SL}} = \frac{10730}{9185} = 1.17$$

High-pressure rockets will have a lower variation of thrust with altitude (atmospheric pressure) than low pressure rockets.

Figure 8 emphasizes the fact that rocket exhaust nozzles attain a high effectiveness only when low ratios of outside pressure to combustion chamber pressure are maintained.

Maximum Altitude

An extension of the fundamental relations developed in the foregoing sections allows the determination of the maximum attainable altitude of rockets launched vertically from the ground. The basic case of a rocket operating in a vacuum is considered first and then rocket projectiles operating in the atmosphere are treated. The atmospheric density at high altitudes was estimated from an extrapolation of NACA standard atmosphere. Since the air resistance at extreme altitudes is very small compared with the force of gravity, the altitudes attained would be changed a negligible amount by an error in assumed air density.

Rocket operating in a vacuum.- The resultant force (thrust minus instantaneous weight) on a rocket accelerating vertically upward in a vacuum can be equated to the mass times the acceleration:

$$V_j \frac{dm}{dt} - \left(m_o - t \frac{dm}{dt}\right)g = \left(m_o - t \frac{dm}{dt}\right)a \tag{29}$$

If the thrust is some constant K times the initial gross mass then, from equation (1),

$$\frac{dm}{dt} = \frac{Km_o}{V_j}$$

and hence

$$a = \frac{KV_j}{V_j - Kt} - g \tag{30}$$

Since

$$\int_0^{V_{inst}} dV = \int_0^t a\ dt = KV_j \int_0^t \frac{dt}{V_j - Kt} - g \int_0^t dt$$

$$V = V_j \log_e\left(\frac{V_j}{V_j - Kt}\right) - gt \tag{31}$$

The combustion time t_B can be expressed in terms of fuel load m_f, jet velocity V_j, and K:

$$m_f = t_B \frac{dm}{dt} = \frac{t_B K m_o}{V_j}$$

or

$$t_B = \frac{V_j}{K} \frac{m_f}{m_o} \tag{32}$$

By use of equation (32), equation (31) becomes

$$V_B = V_j \log_e\left(\frac{1}{1 - \frac{m_f}{m_o}}\right) - g\frac{V_j}{K}\frac{m_f}{m_o} \tag{33}$$

where V_B is the velocity at the end of burning. Figure 9 shows a plot of this equation for $V_j = 8000$ and various values of K/g and m_f/m_o. This graph shows that the rockets with high initial accelerations and high percentages of fuel reach the highest velocities when fired vertically in a vacuum.

In order to determine the altitude attained during burning, equation (31) must be integrated:

$$\int_0^{h_{inst}} dh = \int_0^t V\,dt = V_j \int_0^t \log_e\left(\frac{V_j}{V_j - Kt}\right) dt - g\int_0^t t\,dt$$

Integration of this equation yields

$$h_{inst} = \left(\frac{V_j^2}{K} - V_j t\right)\log_e\left(\frac{V_j - Kt}{V_j}\right) + V_j t - \frac{gt^2}{2} \tag{34}$$

If the value of combustion time from equation (32) is substituted in equation (34), the altitude at the end of burning becomes

$$h_B = \frac{V_j^2}{K}\left[\left(1 - \frac{m_f}{m_o}\right)\log_e\left(1 - \frac{m_f}{m_o}\right) + \frac{m_f}{m_o} - \frac{g}{2K}\left(\frac{m_f}{m_o}\right)^2\right] \tag{35}$$

Figure 10 shows the altitude at the end of burning for the rockets considered in figure 9. The rockets with a high percentage of fuel load and a thrust equal to approximately 1.5 times the starting weight reach the highest altitudes during burning.

After the burning is completed, the rockets coast with a deceleration due to gravity. Since some of the altitudes reached during coasting are very high, the variation with altitude of the acceleration due to gravity must be accounted for. The acceleration of the rocket is

$$\frac{d^2h}{dt^2} = -g_o\left(\frac{r_o}{r_o + h}\right)^2 \tag{36}$$

where

g_o = acceleration of gravity at surface of earth

= 32.2 feet per second^2

and

r_o = radius of the earth

= 20,908,800 feet

Equation (36) can be written

$$V\,\frac{dV}{dh} = -g_o\left(\frac{r_o}{r_o + h}\right)^2 = -\frac{g_o}{\left(1 + \frac{h}{r_o}\right)^2}$$

and

$$\int_0^{V_B} V\,dV = -g_o\int_{h_f}^{h_B}\frac{dh}{\left(1 + \frac{h}{r_o}\right)^2}$$

where h_f is the final altitude reached (when $V = 0$). Integration and simplification of this equation gives

$$h_f = \frac{2\left(r_o + h_B\right)r_o^2 g_o}{2r_o^2 g_o - V_B^2\left(r_o + h_B\right)} - r_o$$

or

$$h_f = \frac{2r_o^2 g_o h_B + V_B^2 r_o\left(r_o + h_B\right)}{2r_o^2 g_o - V_B^2\left(r_o + h_B\right)} \tag{37}$$

The total altitude attained by rockets fired vertically in a vacuum can be determined by use of the values of V_B and h_B determined from equations (33) and (35). Figure 11 shows that the maximum possible altitudes result from high initial accelerations combined with large fuel loads.

Rocket operating in air.- The most important effect of air on the performance of most rockets is the deceleration caused by air resistance. For a rocket of given density, the deceleration due to air resistance decreases as the size of the rocket increases. Since the purpose of this part of the investigation is to show how air resistance shifts the trends that were shown in figure 11 for operation in a vacuum, an estimate of the drag of a possible rocket design must be made. Figure 12 shows the assumed variation of drag coefficient with Mach number. In actual practice the drag would, of course, be a function of Reynolds number too. An additional term must be added to equation (30) to account for the deceleration due to air resistance. The acceleration during burning, therefore, is

$$a_B = \frac{\left(K - \frac{D}{m_o}\right)V_j}{V_j - Kt} - g \tag{38}$$

Equation (38) was used in a graphical integration of the equation

$$h_B = \int_0^{t_B} \int_0^{t} a_B \, dt \, dt$$

where the upper limit t_B was determined by equation (32). After the fuel is consumed, the equation for acceleration during coasting becomes

$$a_c = - \frac{D}{m_e} - g$$

where g varies with altitude. The final altitude thus attained is

$$h_f = h_B + h_c$$

or, where h_c is determined graphically,

$$h_f = h_B + \int_{t_B}^{t_f} V_B \, dt + \int_{t_B}^{t_f} dt \int_{t_B}^{t} a_c \, dt$$

At high altitudes, where the drag becomes negligible, equation (37) can be adapted for use in the final part of the integration.

Figure 13 shows the maximum altitudes reached by rockets fired vertically through the atmosphere. A sketch is given to illustrate the size of the rocket. The altitudes reached are less than those reached in a vacuum (see fig. 11) and indicate that an optimum initial acceleration which is of the order of twice the acceleration of gravity exists for high-altitude rockets.

Possibly the most interesting conclusion to be drawn from figure 13 is that projectiles having a ratio of thrust to initial weight of 100 and a 60-percent fuel load reach higher altitudes than those projectiles with more fuel. This condition is easily explained. Because of the high thrust, these high initial acceleration rockets burn out at low altitudes and rely on coasting to reach high altitudes. The rockets with low empty weights (high design fuel loads) decelerate rapidly because of air resistance and hence do not coast as high as the rockets with heavier empty weights. If the air resistance is considered as a function of Reynolds number the intersections will differ.

Escape velocity.- Equation (37) gave the equation for the maximum altitude reached by a rocket in a vacuum as

$$h_f = \frac{2\left(r_o + h_B\right)r_o^2 g_o}{2r_o^2 g_o - V_B^2\left(r_o + h_B\right)} - r_o$$

or

$$h_f = \frac{2r_o^2 g_o h_B + V_B^2 r_o\left(r_o + h_B\right)}{2r_o^2 g_o - V_B^2\left(r_o + h_B\right)}$$

This altitude becomes infinite when

$$2r_o^2 g_o - V_B^2\left(r_o + h_B\right) = 0$$

Then if h_B is negligibly small compared with r_o

$$V_B = \sqrt{2g_o r_o} \tag{39}$$

$$= 36,695 \text{ feet per second}$$

$$= 6.95 \text{ miles per second}$$

This velocity is called "escape velocity." If the effect of air resistance is neglected a projectile having this speed could completely escape the gravitational field of the earth.

The percentage of fuel load required to reach escape velocity can be determined for rockets which have high accelerations by use of equations (4) and (39), thus

$$\frac{W_f}{W_o} = 1 - e^{-\frac{\sqrt{2g_o r_o}}{V_j}} = 1 - e^{-\frac{36695}{V_j}} \tag{40}$$

Figure 14 shows a graph of the percentage fuel load required to reach escape velocity with different effective jet velocities. Curves are presented for one-stage, two-stage, and three-stage rockets.

The performance of a multistage rocket can be explained by referring to equation (4) which shows that the change in speed of a rocket is a function of the ratio of fuel weight to gross weight. A two-stage rocket can be considered to consist of a "mother" rocket and a "baby" rocket. The combined rocket is fired starting from rest. The weight of the fuel in the mother rocket divided by the gross weight of the combined rocket can be used in equation (4) to determine the final velocity of the combined rocket. When this speed is reached all the fuel in the mother rocket has been used but none of that in the baby rocket has been used. At this time the mother rocket is cut loose from the baby rocket and the engine of the baby rocket is started. Then the weight of fuel of the baby rocket divided by the gross weight of the baby rocket determines the increase in speed of the second stage of the rocket. (See equation (4).) If the ratio of fuel weight of the mother rocket to gross weight of the combination is the same as the fuel weight of the baby rocket divided by the initial weight of the baby rocket, the final velocity of the rocket is twice the velocity that could be obtained by either stage alone. This principle is of extreme importance in attaining the exceedingly high speeds that are required by long-range rockets. Structural factors must be considered before the optimum number of stages can be determined for a given mission.

Figure 14 shows that one-stage and two-stage rockets must have extremely high fuel loads in order to reach escape speeds. The fuel loads required by the three-stage rocket, however, are much lower. For example, a three-stage rocket having a 10,000-foot-per-second jet velocity could reach escape speed provided the fuel load per stage is over 71 percent of the initial weight per stage.

Maximum Range

In order to find the range of a rocket launched from the earth in a vacuum the rocket will be considered to be a free body with a given total energy (that is, to leave the earth with a given velocity and receive no further thrust) moving in the gravitational field of the earth. The path of the rocket is first determined from basic considerations and then the range on the surface of the earth is determined from the intersection of this path with the surface of the earth. The equation that describes the path of such a body has been developed in classical mechanics (reference 1) and is now known as Kepler's first law. A very brief outline of the derivation follows: The law of conservation of energy may be written as (Note - the subscript o denotes the reference level of our system; that is, the surface of the earth.)

$$\frac{V^2}{2} - G\frac{M}{r} = \frac{{V_o}^2}{2} - G\frac{M}{r_o} \tag{41}$$

where

G universal constant of gravitation

M mass of earth

and the law of conservation of angular momentum is, in polar coordinates,

$$r^2 \frac{d\phi}{dt} = \text{Constant} = C \tag{42}$$

The following polar equation is obtained by eliminating time from equations (41) and (42), separating variables, integrating, and simplifying:

$$r = \frac{\frac{C^2}{GM}}{1 - \cos \phi \sqrt{\left(\frac{r_o {V_o}^2}{GM} - 2\right)\frac{C^2}{r_o GM} + 1}} \tag{43}$$

This equation is a special case of Kepler's first law which states that the orbits of bodies in the solar system are conics with the sun occupying one focus.

For the energy levels considered here, the conic is an ellipse with the center of the earth occupying one focus. If the intersections of a circle of radius equal to the radius of the earth with ellipses corresponding to various projectile energy levels are determined, the variation of range with launching angle and total energy can be defined.

In equation (43) when $r = r_o$ = radius of the earth, the ellipse intersects the surface of the earth.

Since

$$C = r_oV_o \cos\theta \tag{44}$$

where θ , the launching angle for the rocket, is the angle between slope of ellipse and slope of circle at point of intersection, equation (43) now becomes

$$r_o = \frac{\dfrac{r_o^2V_o^2\cos^2\theta}{GM}}{1 - \cos\phi\sqrt{1 - \left(2 - \dfrac{r_oV_o^2}{GM}\right)\dfrac{r_oV_o^2}{GM}\cos^2\theta}} \tag{45}$$

where ϕ is the semiangle of the range as measured from the center of the earth.

If

$$Q = \frac{r_oV_o^2}{GM} = \left(0.3201 \times 10^{-8}\right)V_o^2 \tag{46}$$

where V_o is measured in miles per hour, then

$$1 - \cos\phi\sqrt{1 - (2 - Q)Q\cos^2\theta} = Q\cos^2\theta$$

or

$$\cos\phi = \frac{1 - Q\cos^2\theta}{\sqrt{1 - (2 - Q)Q\cos^2\theta}} \tag{47}$$

Also

$$\sin\phi = \frac{Q\sin\theta\cos\theta}{\sqrt{1 - (2 - Q)Q\cos^2\theta}} \tag{48}$$

and

$$\tan \phi = \frac{Q \sin \theta \cos \theta}{1 - Q \cos^2\theta} \tag{49}$$

If the effect of the rotation of the earth is neglected

$$R = \frac{2\pi r_o}{180}\phi$$

where ϕ is measured in degrees, or

$$R = 138.24 \tan^{-1} \frac{Q \sin \theta \cos \theta}{1 - Q \cos^2\theta} \tag{50}$$

where R is measured in miles. This expression for R is the idealized range obtained when air drag and burning time are neglected.

Figure 15 shows the variation with launching angle of the range of a rocket launched in vacuum for constant values of launching velocity.

In order to determine the launching angle for maximum range at a given launching velocity, equation (50) is differentiated with respect to θ and the result set equal to zero so that

$$\begin{aligned} \theta_{opt} &= \tan^{-1} \sqrt{1 - Q} \\ &= \tan^{-1} \sqrt{1 - \left(0.3201 \times 10^{-8}\right)V_o^2} \end{aligned} \tag{51}$$

Figure 15 shows that the optimum launching angle for short-range rockets is 45°, however, this optimum launching angle decreases to 0° as the launching velocity is increased to that required for a range of one-half the circumference of the earth.

When the value of Q from equation (51) is substituted in equation (50),

$$R_{max} = 138.24 \sin^{-1} \frac{V_o^2}{(6.248 \times 10^8) - V_o^2} \tag{52}$$

This maximum range is plotted in figure 16 as a function of launching velocity. This figure shows that a rocket having a speed of 17,800 miles per hour can reach any point on the surface of the earth. Since this rocket skims the surface of the earth (effects of the atmosphere neglected), it can be concluded that any point on the surface of the earth can be reached by this rocket in less than 45 minutes.

The maximum altitude attained by a rocket launched at the optimum launching angle for the launching velocity considered may be determined as follows: In equation (43) the maximum value of r occurs at $\phi = 0^\circ$ (that is, $\cos \phi = 1$). If we substitute this value of $\cos \phi$ in equation (45) we obtain by use of equation (46)

$$r = r_o \frac{Q \cos^2\theta}{1 - \sqrt{1 - Q(2 - Q) \cos^2\theta}} \tag{53}$$

If the value of θ_{opt} is substituted in equation (53) and the radius of the earth subtracted from r, the following equation results:

$$h_{max} = r_o \left(\frac{1 + \sqrt{1 - Q}}{2 - Q} - 1\right) \tag{54}$$

A plot of equation (54) is presented as figure 17.

Equation (52) determines the range of a rocket as a function of its launching velocity. Thus, for any range, the required launching speed can be obtained and then substituted in equation (4) to determine the fuel loads needed to reach the required speed. Figure 18 shows the fuel loads required to attain different ranges for several values of the jet velocity and for one-stage, two-stage, and three-stage rockets. For example, a three-stage rocket having a 10,000-foot-per-second jet velocity can reach any point on the surface of the earth provided the fuel load per stage is over 58 percent of the initial weight per stage (fig. 18(c)).

The range charts as presented in this paper consider nonlifting aircraft operating in a vacuum. This method of operation can be approximated by boosting the rocket through most of the atmosphere, by firing it from a high-flying airplane, or by the use of extremely large, high-density rockets in which the ratio of air drag to rocket mass is fairly low.

The air resistance in some cases may appreciably slow down the rocket and hence shorten its range as a projectile. On the other hand, the use of wings on a rocket that reenters the atmosphere may regain much of the lost range by enabling the rocket to glide for a considerable distance.

The problem of range in the atmosphere may be demonstrated by the use of two similar rockets having three stages, a fuel load per stage of 60 percent of the initial weight per stage, and an effective jet velocity of 8000 feet per second. Figure 18(c) shows that a rocket having these characteristics should be capable of having a range of 4700 miles in a vacuum.

The determination of the optimum flight path and range of rockets operating in air is beyond the scope of this paper; however, certain assumptions can be made for a rocket similar to the preceding three-stage rocket. If the rocket is assumed to rise vertically through the atmosphere to an altitude of 100,000 feet and to use the remaining fuel to travel as a projectile in a vacuum, the performance can be read from the graphs. Figure 13 shows that a 42-percent fuel load is required to reach 100,000 feet. The remaining fuel in the first stage is then

$$\frac{W_f - W_{fB}}{W_o - W_{fB}} = \frac{W_{fR}}{W_R} = \frac{0.60 - 0.42}{1.00 - 0.42} = 0.31$$

If a 31-percent fuel load is used for the first stage and a 60-percent fuel load for the last two stages, figure 2 gives the final speed of the rocket as

$$\begin{aligned} V_f &= 2025 + 2(4975) \\ &= 11{,}975 \text{ miles per hour} \\ &= 17{,}567 \text{ feet per second} \end{aligned}$$

Figure 16 gives an estimate of the range for travel at an altitude of more than 100,000 feet as 2400 miles. When the projectile returns to an altitude of 100,000 feet the total energy per pound of rocket weight is

$$\begin{aligned} E = \frac{V^2}{2g} + h &= \frac{17567^2}{64.4} + 100{,}000 \\ &= 4{,}892{,}000 \text{ foot-pounds} \end{aligned}$$

If the rocket can glide with an average effective lift-drag ratio of 3, this energy can be translated into the additional range (final energy neglected)

$$\Delta R = \frac{3E}{5280} = 2780 \text{ miles}$$

The total range in the atmosphere would therefore be

$$R = 2400 + 2780 = 5180 \text{ miles}$$

This example shows the significant gains in range which may be obtained by use of gliding devices on rocket projectiles for operation in air and the importance of securing good gliding characteristics for operation in the air. Some other important problems involved in attaining maximum possible range are the determination of the optimum flight paths, the investigation of heating caused by the air, a study of the loads imposed by leveling out from the trajectory, and a suitable design for high lift-drag ratios at high Mach numbers.

CONCLUDING REMARKS

The present paper has summarized some of the existing literature on rocket theory and has supplemented this information with additional interesting facts and relations useful in introducing the problems of rocket-powered aircraft performance. Some general and specific facts relative to rocket performance brought out are as follows:

1. The jet efficiency of a rocket accelerating horizontally in a vacuum reaches a maximum of 100 percent at a forward speed equal to the jet speed and then decreases to zero at twice jet speed.

2. A rocket has its maximum kinetic energy when it attains a speed equal to twice the jet speed.

3. In a vacuum any rocket attaining a velocity of 17,800 miles per hour could operate in a circular orbit around the earth at ground level thereby being capable of reaching any point on the earth's surface in less than 45 minutes. In practice, the performance estimation must allow for the effects of the atmosphere.

4. If the effects of air resistance are neglected, any point on the surface of the earth can be reached by a three-stage rocket having a 10,000-foot-per-second jet velocity provided the fuel load per stage is over 58 percent of the initial weight per stage.

5. The three-stage rocket would require fuel loads equal to 71 percent of the initial weight per stage in order to escape the gravitational field of the earth.

6. The optimum launching angle for attaining maximum range with a given launching velocity in a vacuum varies linearly from 45° for zero range to 0° for a range equal to one-half the circumference of the earth.

7. Rocket exhaust nozzles attain high effectiveness only when low ratios of outside pressure to combustion chamber pressure are maintained.

Langley Memorial Aeronautical Laboratory
National Advisory Committee for Aeronautics
Langley Field, Va., September 29, 1947

REFERENCE

1. Moulton, Forest Ray: An Introduction to Celestial Mechanics. Second rev. ed., The Macmillan Co., 1928.

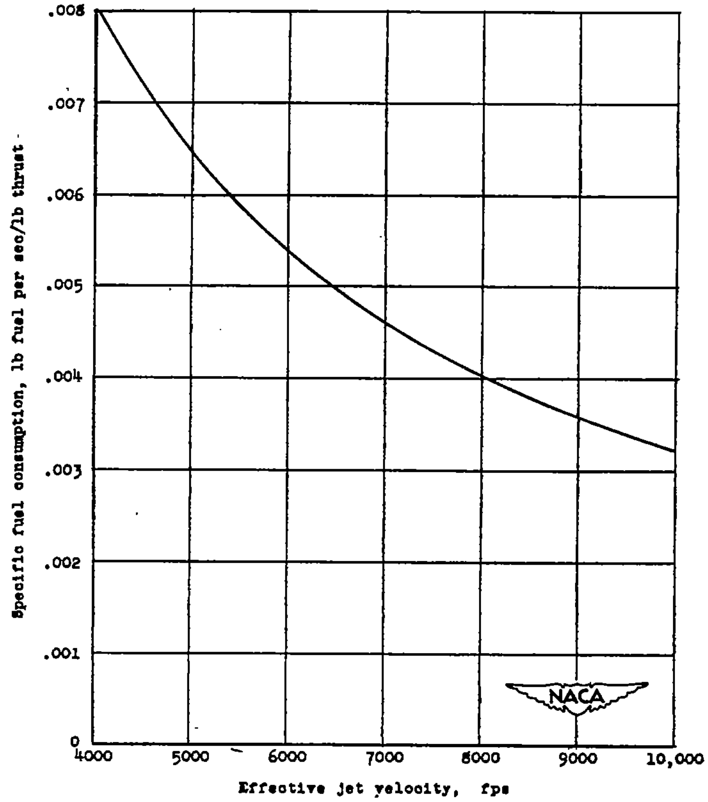


Figure 1.- Specific fuel consumption as a function of jet velocity.

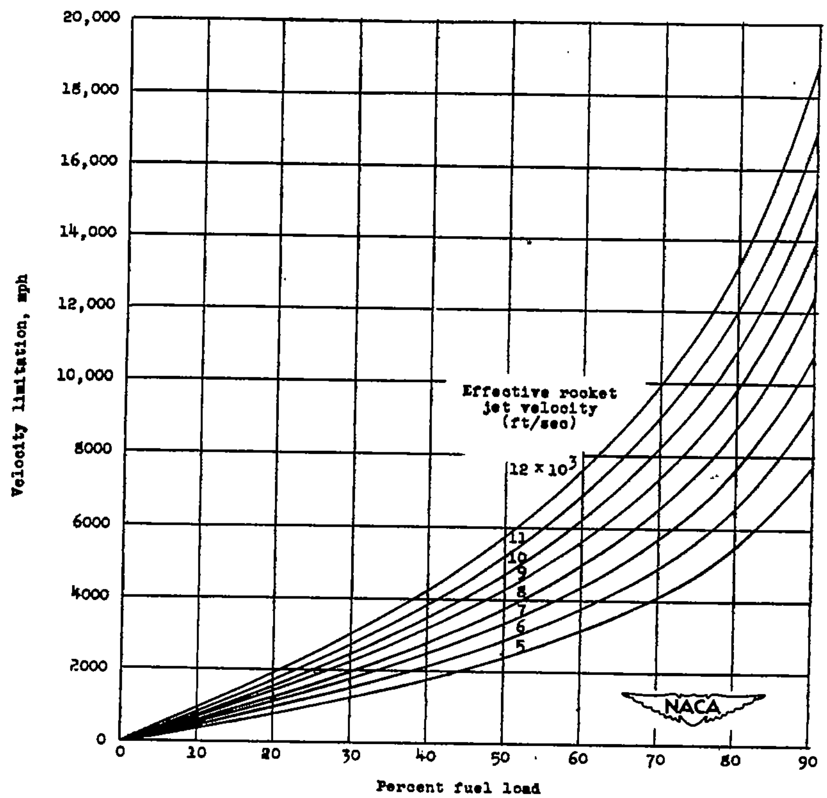


Figure 2.- Velocity limitation set by fuel load and effective jet velocity.

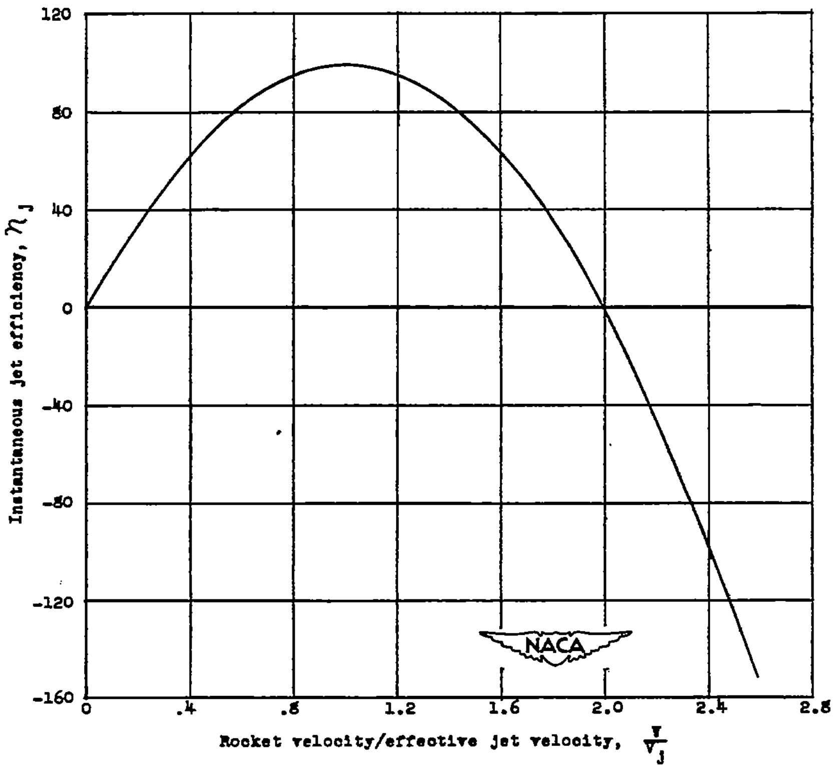


Figure 3.- Instantaneous jet efficiency as a function of the ratio of rocket velocity to effective jet velocity.

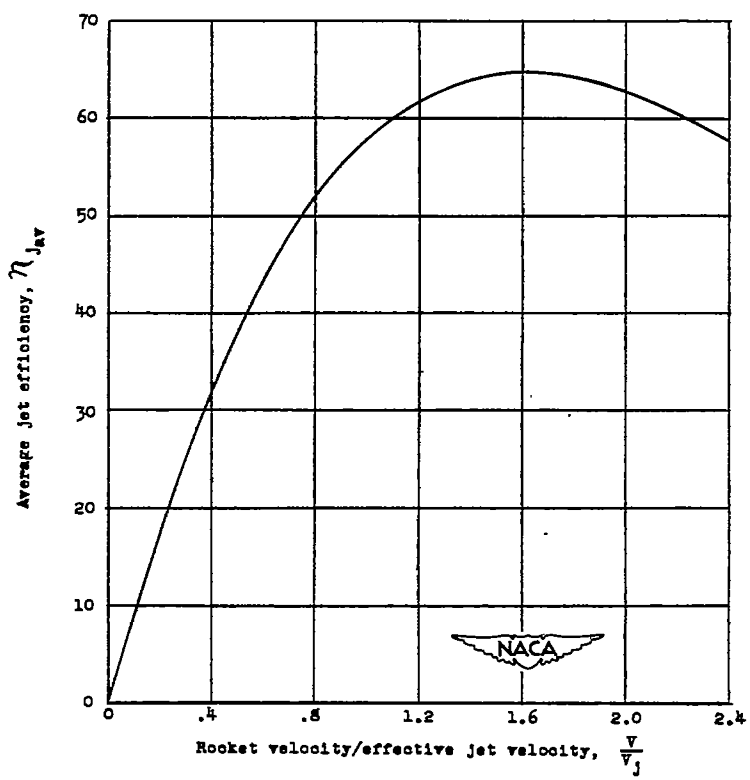


Figure 4.- Average jet efficiency as a function of the ratio of rocket velocity to effective jet velocity.

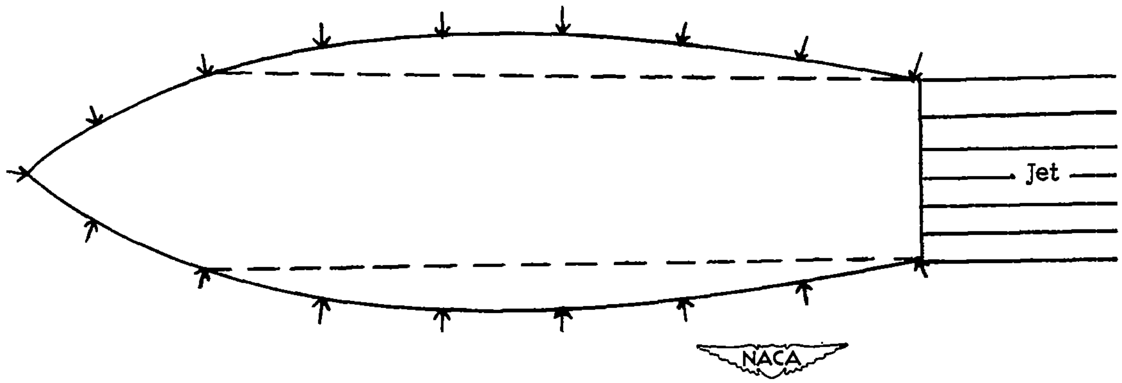


Figure 5.- Forces on outer surface of rocket at rest.

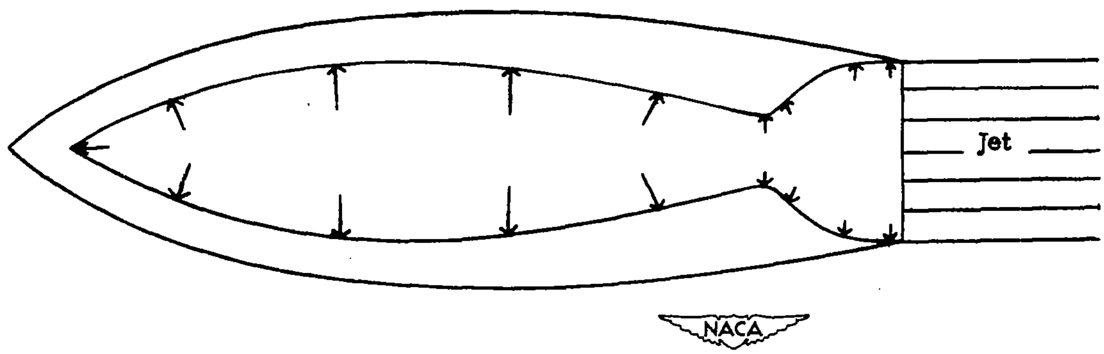


Figure 6.- Forces on inner surface of rocket.

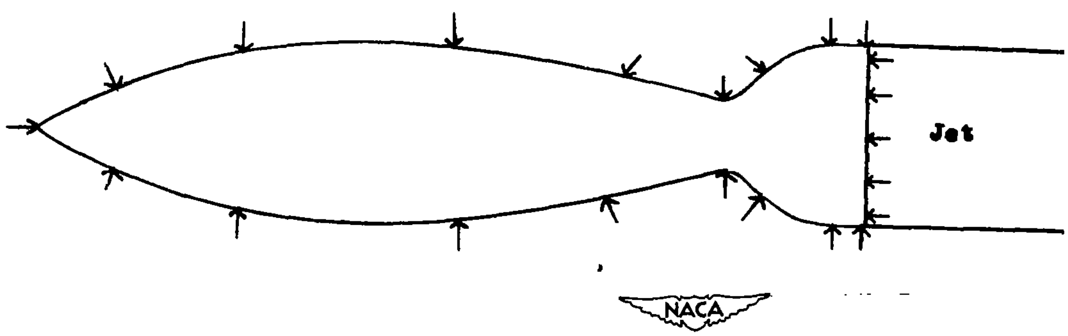


Figure 7.- Forces on burning gases.

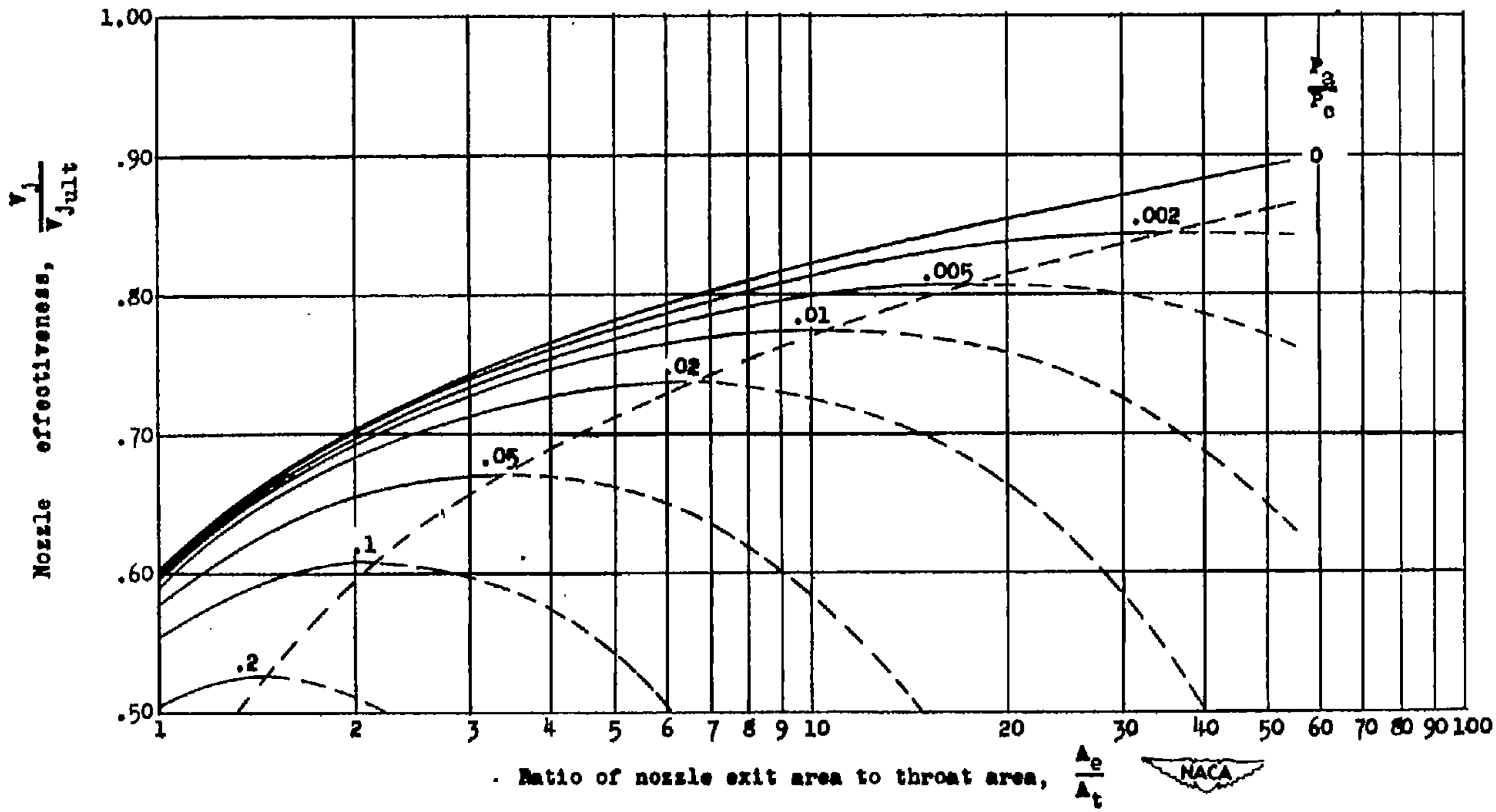


Figure 8.- Effect of ratio of nozzle exit area to throat area on nozzle effectiveness.

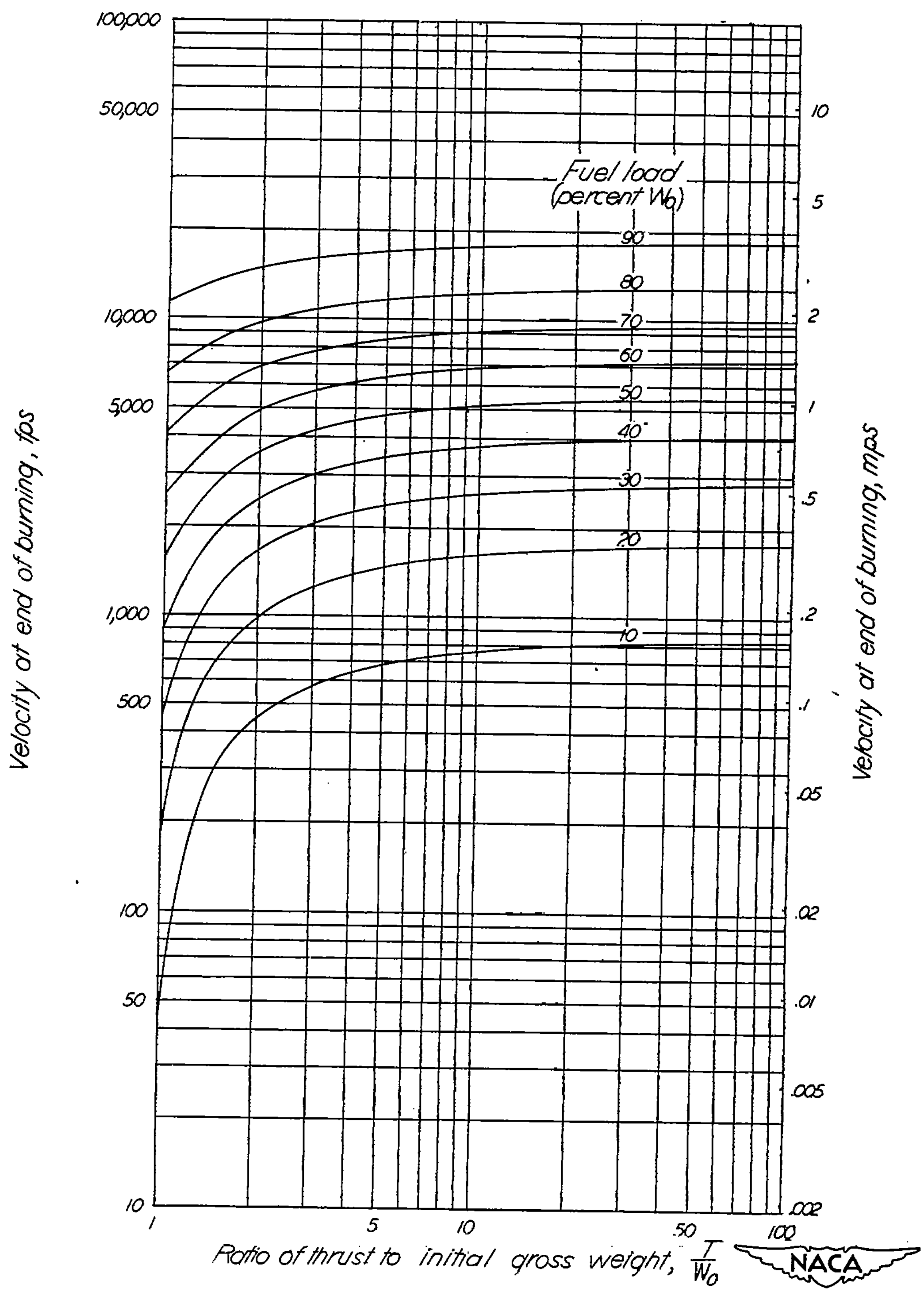


Figure 9.- Velocity at end of burning for rocket projectiles fired vertically in a vacuum. V_j = 8000 feet per second.

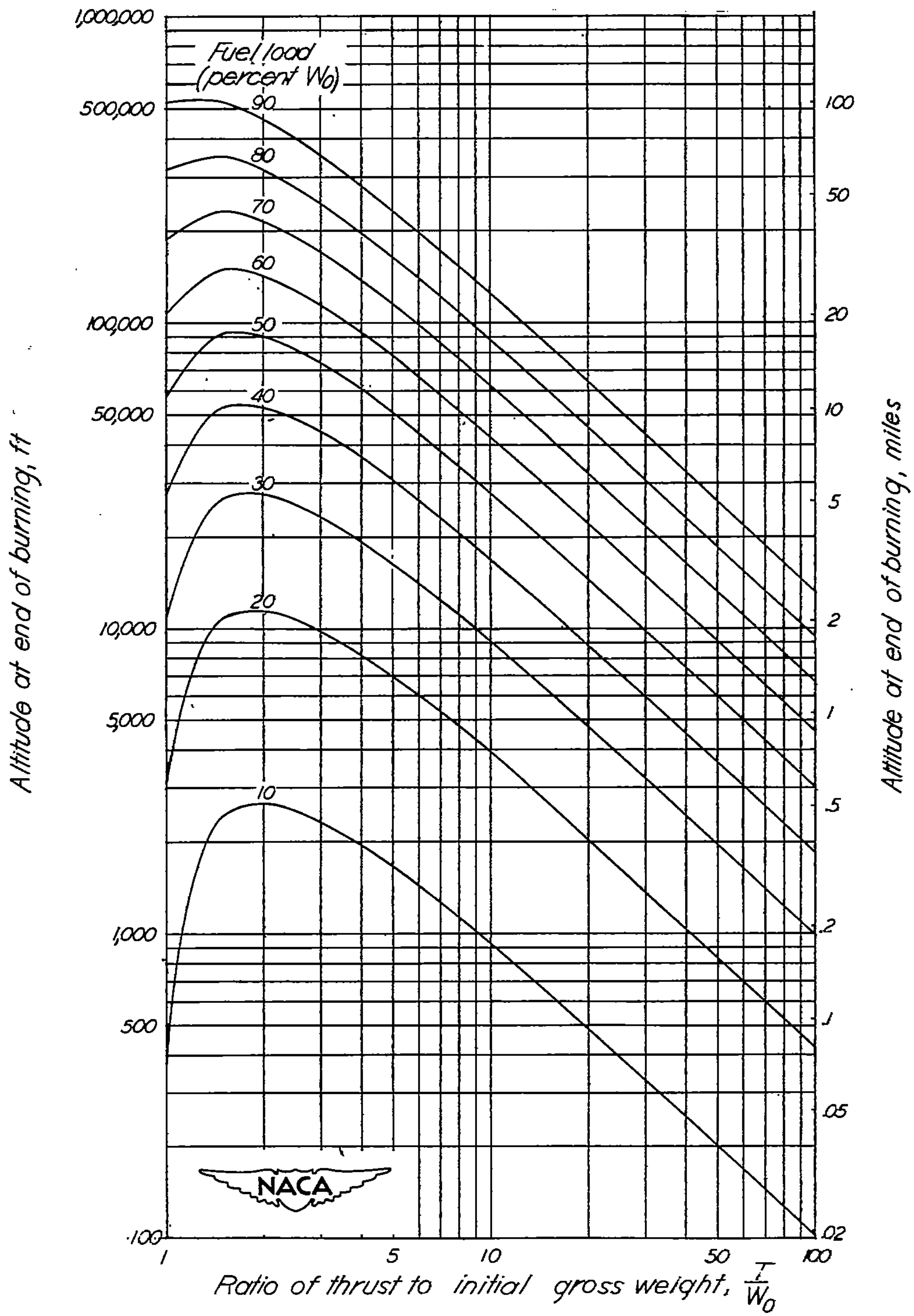


Figure 10.- Altitude at end of burning for rocket projectiles fired vertically in a vacuum. V_j = 8000 feet per second.

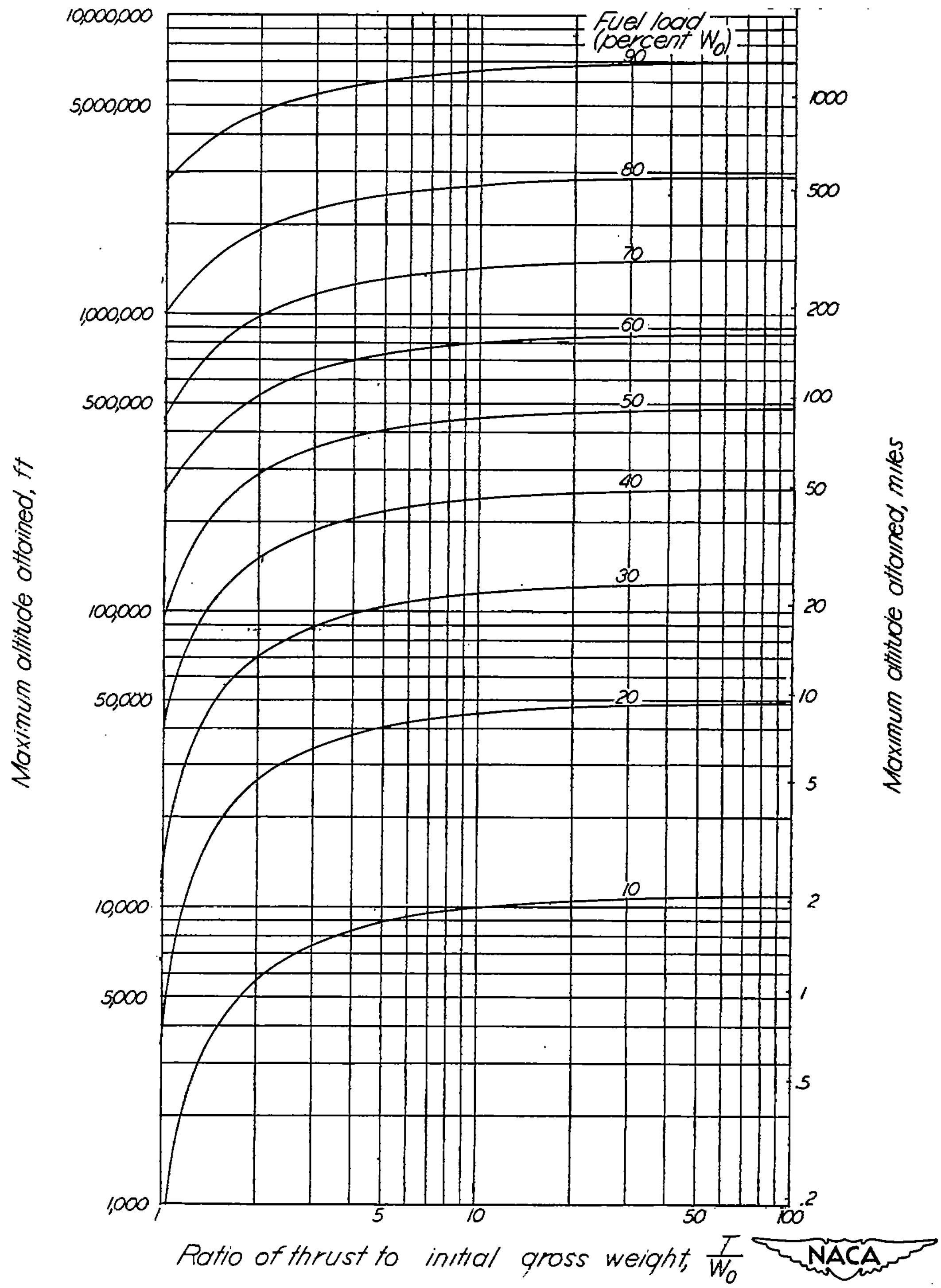


Figure 11.- Maximum altitude of rocket projectiles fired vertically in a vacuum. V_j = 8000 feet per second.

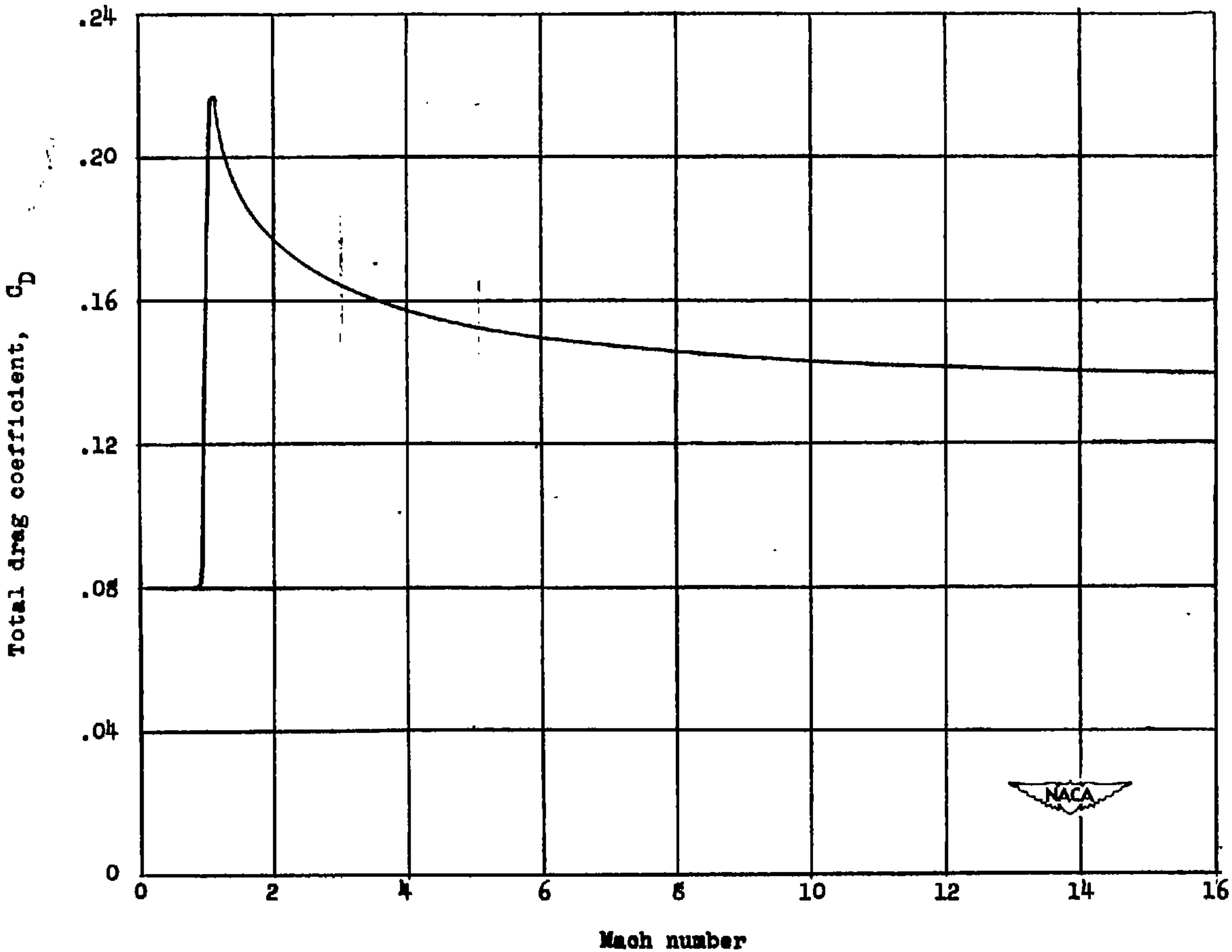


Figure 12.- Assumed variation of drag coefficient with Mach number.

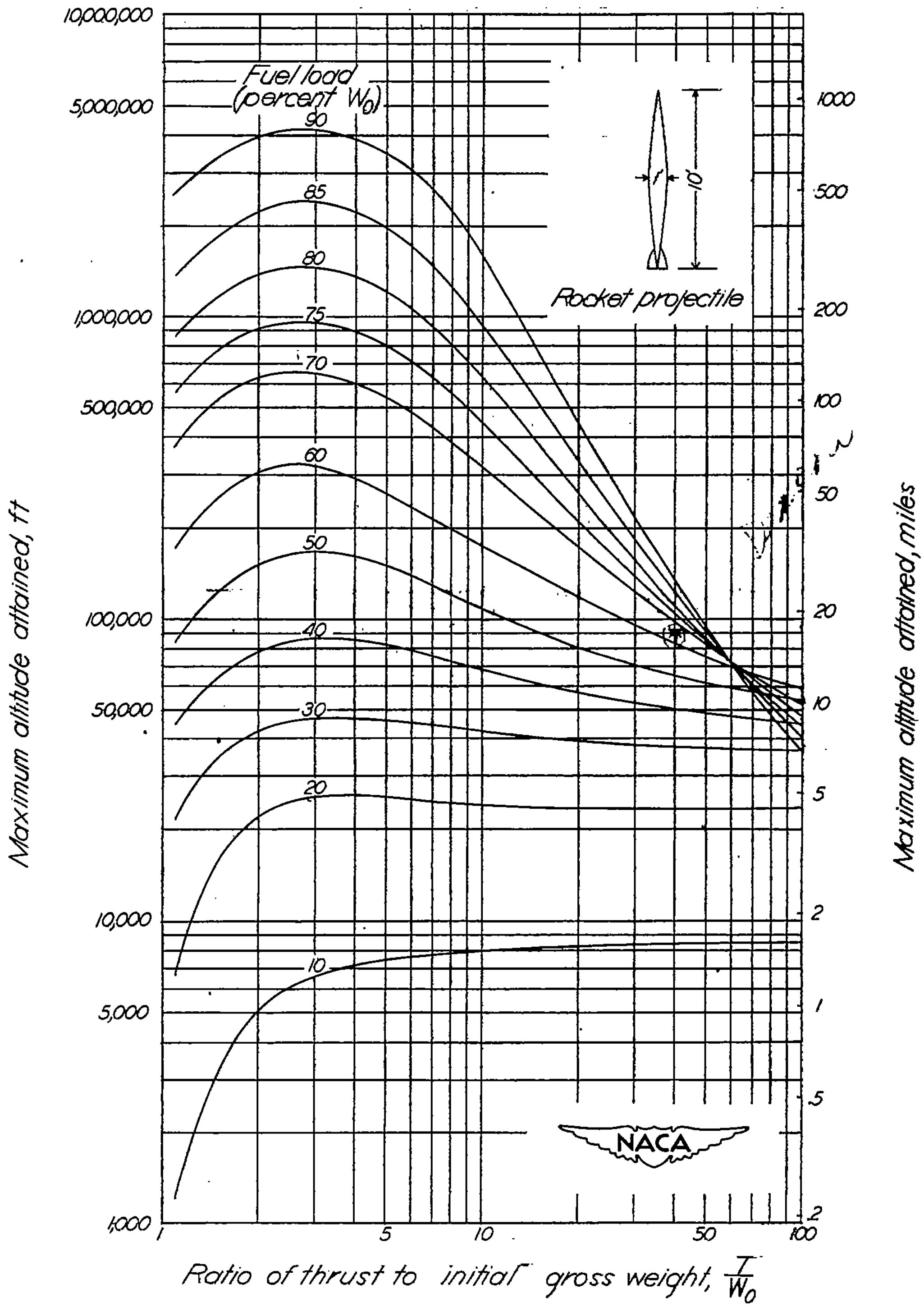


Figure 13.- Maximum altitude of rocket projectiles fired vertically in standard air. V_j = 8000 feet per second.

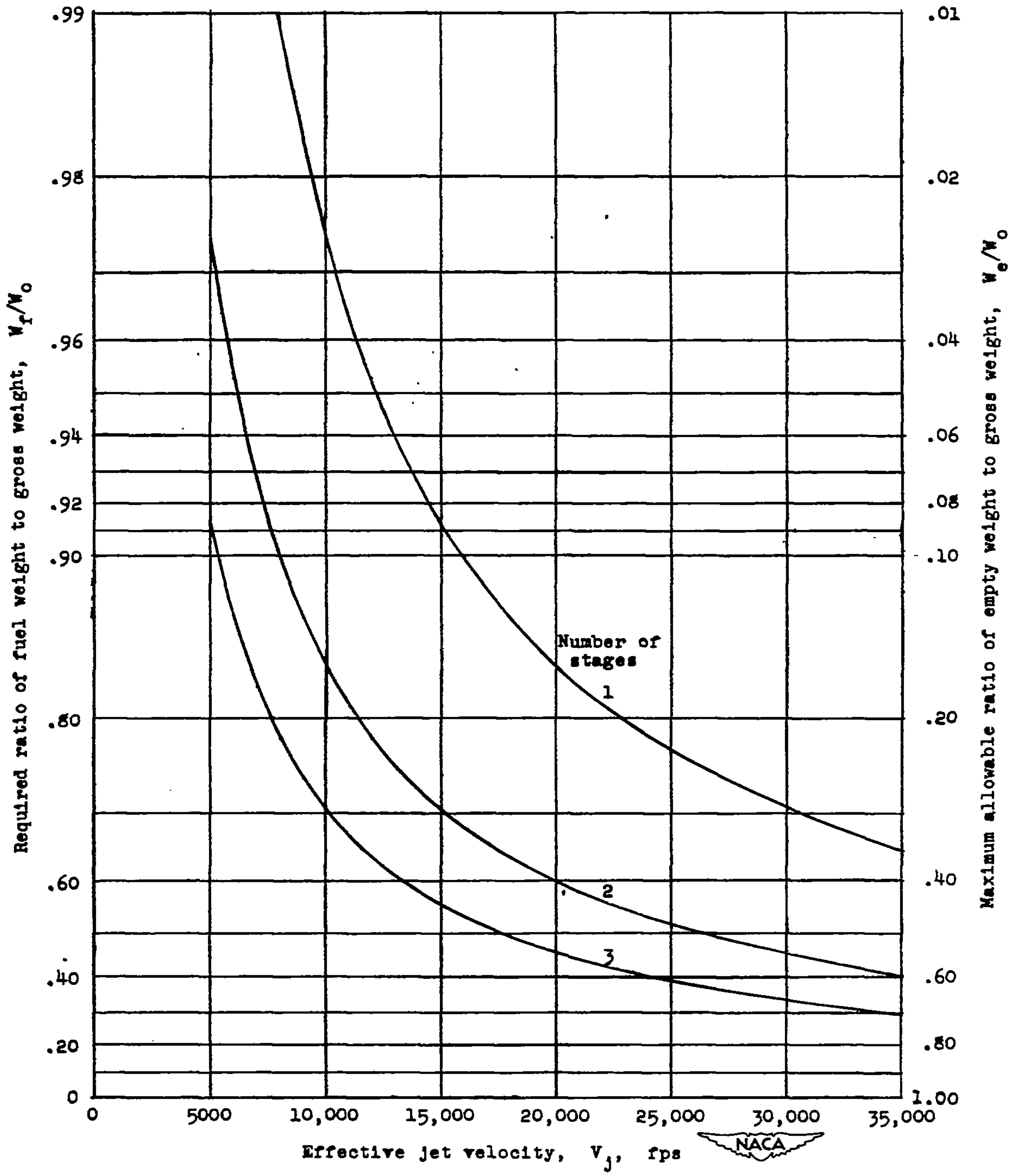


Figure 14.- Weight ratios required for several stages to attain escape velocity for rocket.

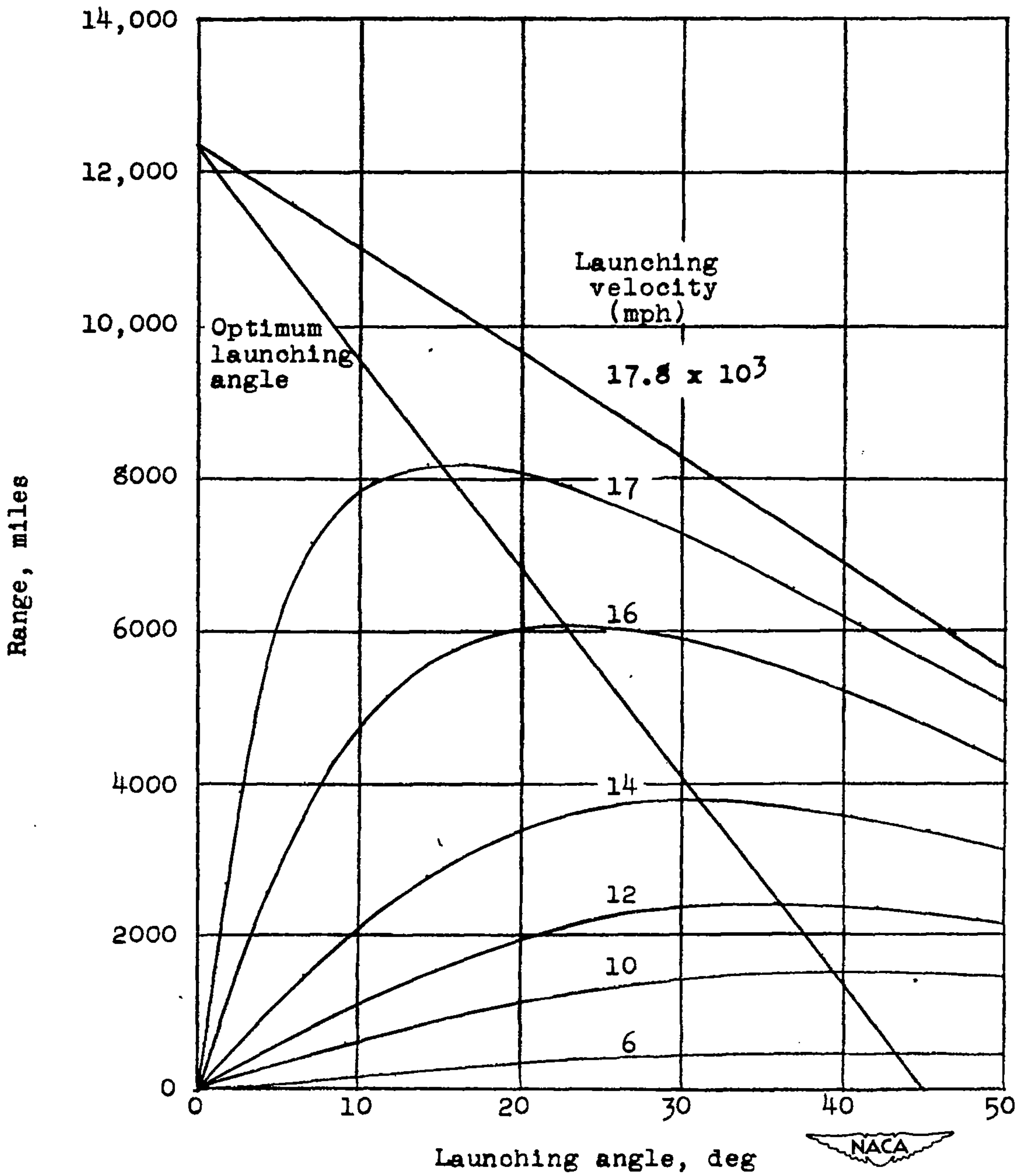


Figure 15.- Variation of range with launching angle for rocket projectiles launched in a vacuum.

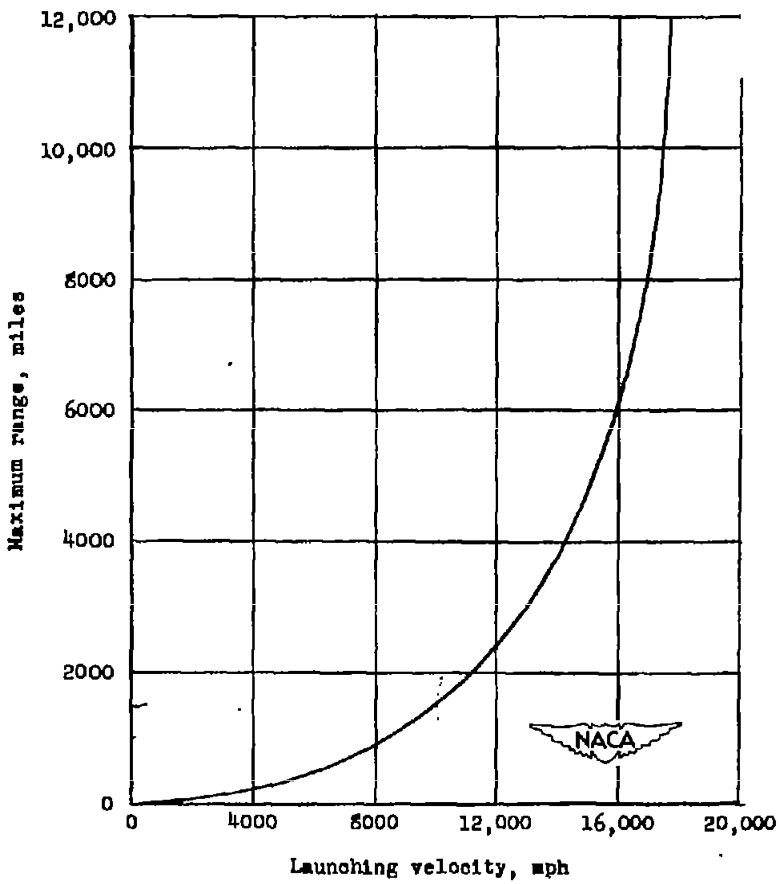


Figure 16.- Variation of maximum range with launching velocity for rocket projectiles launched at optimum angle in a vacuum.

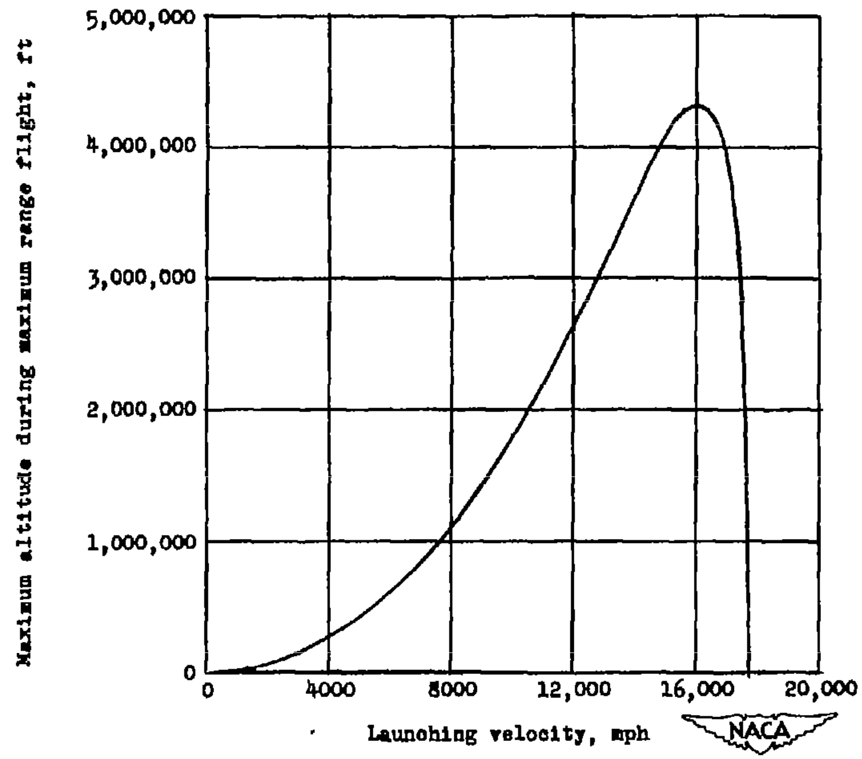
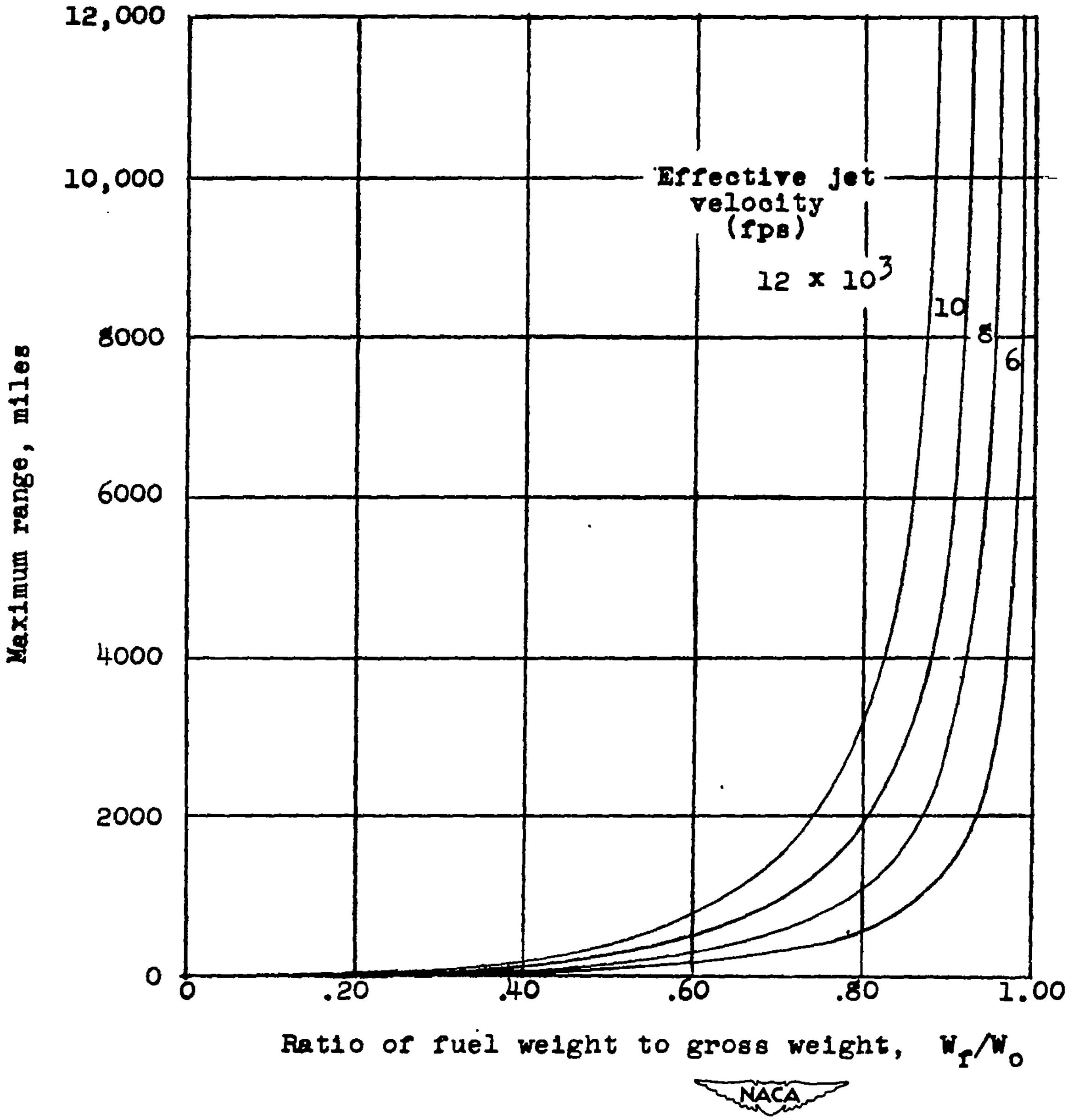
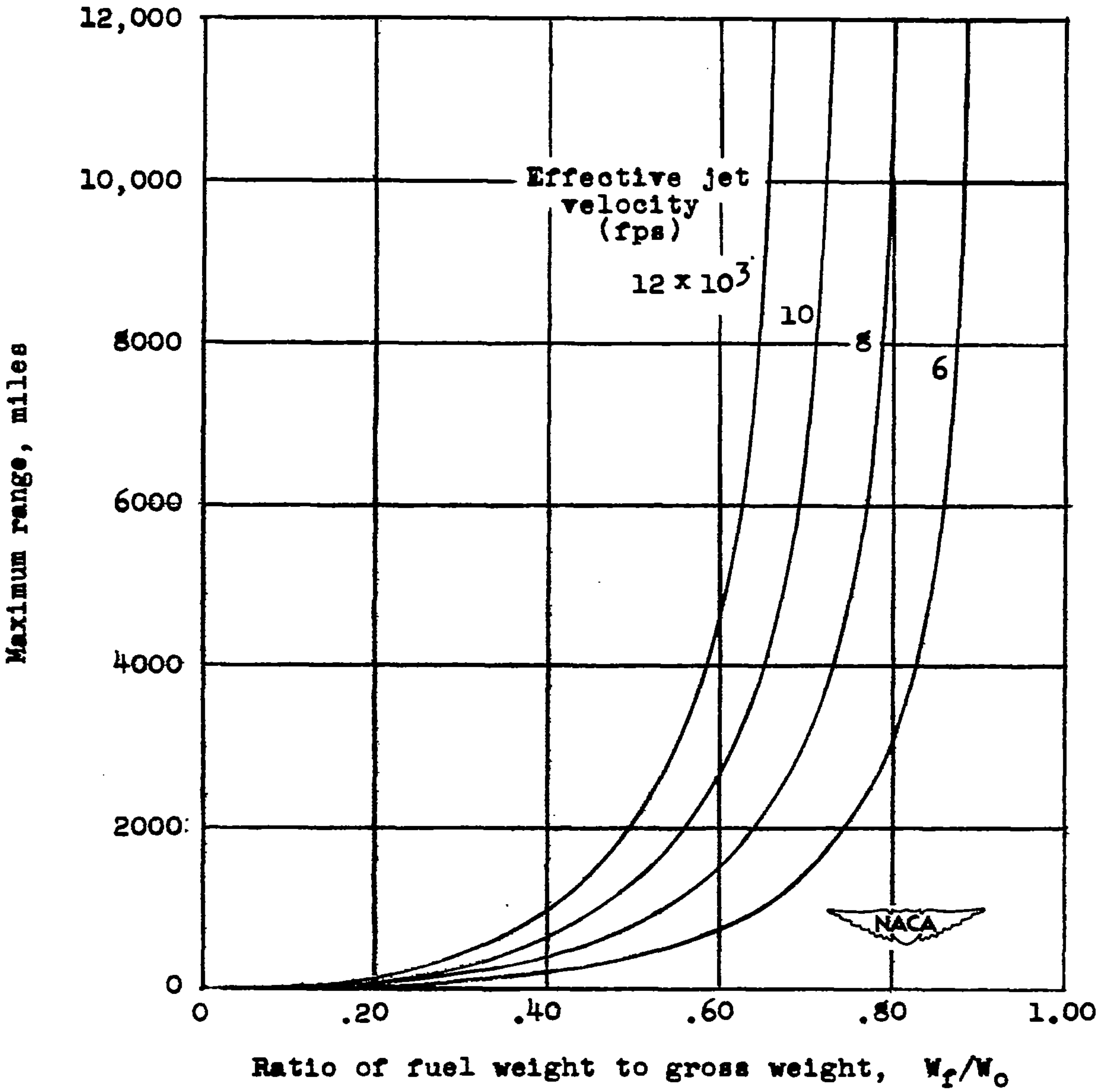


Figure 17.- Maximum altitude attained for rocket projectiles fired at angle for maximum range in a vacuum.



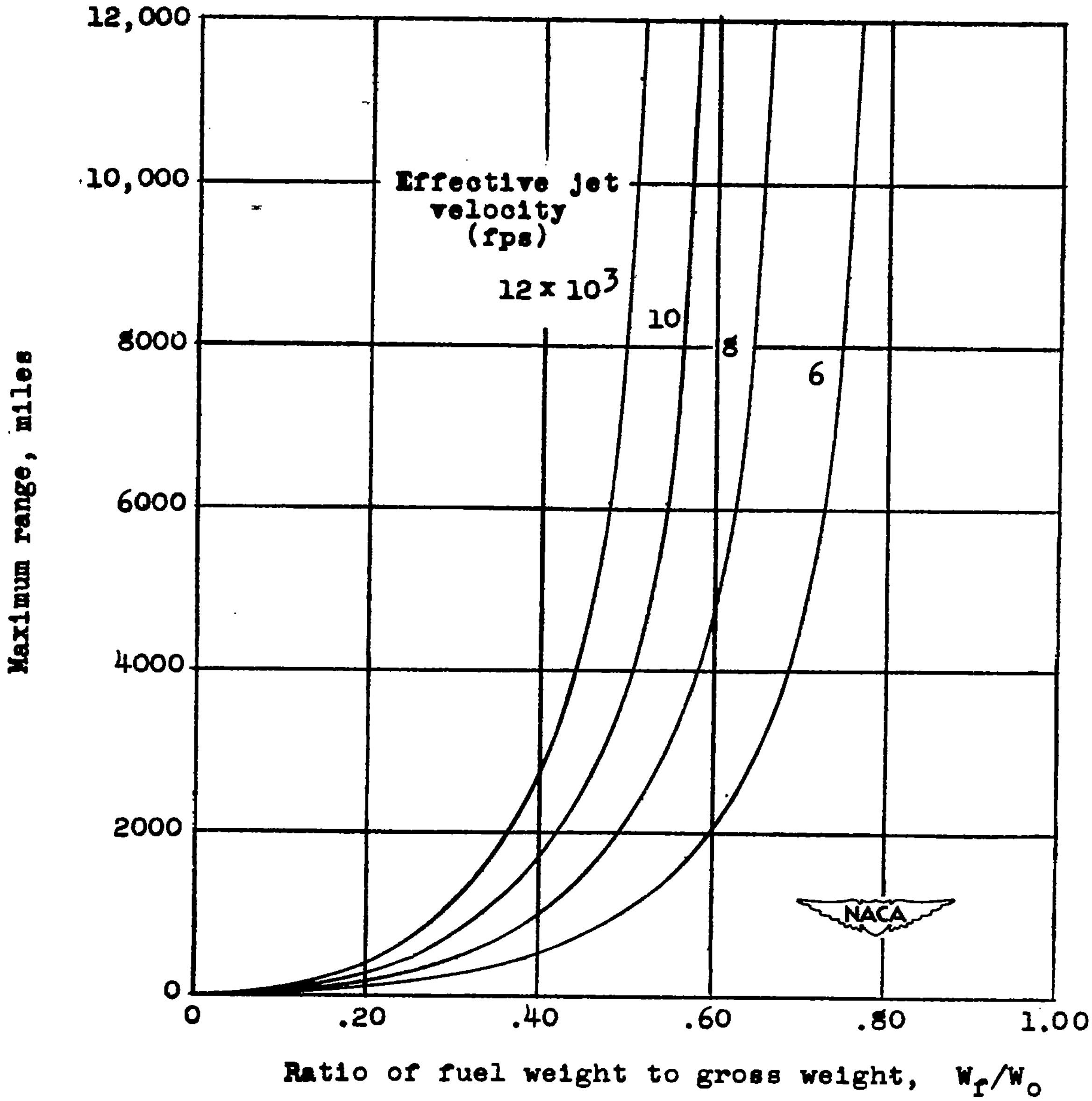
(a) One-stage rocket.

Figure 18.- Maximum range as a function of the ratio of fuel weight to gross weight for several effective jet velocities for rocket projectiles launched in a vacuum.



(b) Two-stage rocket.

Figure 18.- Continued.



(c) Three-stage rocket.

Figure 18.- Concluded.